# Reduction of the Two Body Problem in $N = 2$ Chern Simons supergravity

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**Abstract**

By an extension of the methods used for the reduction of the two body problem in 2+1 dimensional gravity, we show that the two body problem in N=2 Chern Simons supergravity can be reduced exactly to an equivalent one body formalism. We give exact expressions for the invariants of the reduced one body problem.

In a recent work [1,2], it was shown that the two body problem for sources of any spin in 2+1 dimensional gravity can be solved exactly by reducing it to an equivalent one body problem. The work was motivated by an earlier work of 't Hooft [3] who was the first to recognize the possibility that the two boby problem can be solved exactly. To carry out the reduction, essential use was made of the Chern Simons gauge theory of the Poincaré group,the study of which was initiated by Witten [4,5]. This formulation, aside from its relevance to quantization, has the advantage of making the topological properties of the theory manifest. These properties were used to obtain, among other things, the mass and spin associated with equivalent one body problem and the exact expression for the two particle scattering amplitude [2,6].

The main object of this letter is to show that the two body problem in $N = 2$ supergravity theory in 2+1 dimensions is also exactly solvable. Again we make use of the fact that supergravity theories in 2+1 dimensions can also be formulated as Chern Simons gauge theories of appropriate supersymmetric gauge groups [4,5]. Some features of the $N = 2$ theory has been discussed in the literature. The pure $N = 2$ theory has been discussed by Dayi [7]. Its coupling to matter, the corresponding

Wilson loop observables,and application to superparticle scattering in the test particle approximation was considered by Kohler,et al.[8,9]. To establish our notation, we start with a brief summary of the properties of the pure Chern Simons theory as given in references [1] and [2].

The $N = 2$ Poincaré superalgebra in 2+1 dimensions can be written as

$$\begin{array}{lcl} [J^a, J^b] = -i\epsilon^{abc} J_c & ; & [P^a, P^b] = 0 \\ [J^a, P^b] = -i\epsilon^{abc} P_c & ; & [P^a, Q_\alpha] = 0 \\ [J^a, Q_\alpha] = -(\sigma^a)_\alpha^{\ \beta} Q_\beta & ; & [P^a, Q'_\alpha] = 0 \\ [J^a, Q'_\alpha] = -(\sigma^a)_\alpha^{\ \beta} Q'_\beta & ; & \{Q_\alpha, Q_\beta\} = 0 \\ \{Q_\alpha, Q'_\beta\} = -\sigma^a_{\ \alpha\beta} P^a & ; & \{Q'_\alpha, Q'_\beta\} = 0 \\ a = 0, 1, 2 & ; & \alpha = 1, 2 \end{array} \tag{1}$$

The two component spinor charges $Q_\alpha$ and $Q'_\alpha$ are raised and lowered by the antisymmetric metric $\epsilon^{\alpha\beta}$ with $\epsilon^{12} = -\epsilon_{12} = 1$. the $SO(1,2)$ matrices $\sigma^a$ satisfy the Clifford algebra

$$\{\sigma^a, \sigma^b\} = \frac{1}{2}\eta^{ab} \tag{2}$$

where $\eta^{ab}$ is the Minkowski metric with signature $(+, -, -)$. We also have

$$\sigma^a_{\ \alpha\beta} = (\sigma^a)_\alpha^{\ \gamma} \epsilon_{\gamma\beta} \tag{3}$$

A typical Chern Simons action in 2+1 dimensions is given by

$$I_{cs} = \int_M \gamma_{BC} A^B \wedge (dA^C + \frac{1}{3} f^C_{\ DE} A^D \wedge A^E) \tag{4}$$

where $A^B$ are the components of the Lie superalgebra valued connection

$$A = A^B G_B \quad ; \quad A^B = A^B_\mu dx^\mu \tag{5}$$

The quantity $\gamma$ is a suitable non-degenerate metric on the superalgebra. For the Poincaré superalgebra, its non-zero components are given by [5]

$$< J^a, P^b >=< P^a, J^b >= \eta^{ab} \ ; \ < Q^\alpha, Q'^\beta >=< Q'^\alpha, Q^\beta >= \epsilon^{\alpha\beta} \tag{6}$$

For this algebra, the connection can be written as

$$A_\mu = e_\mu^{\ a} P_a + \omega_\mu^{\ a} J_a + \chi_\mu^{\ \alpha} Q_\alpha + \xi_\mu^{\ \alpha} Q'_\alpha \tag{7}$$

Then with the covariant derivative

$$D_\mu = \partial_\mu + iA_\mu \tag{8}$$

The components of the field strength tensor are given by

$$
\begin{aligned}
F_{\mu\nu} &= -i[D_\mu, D_\nu] \\
&= P_a[\partial_\mu e_\nu^{\ a} - \partial_\nu e_\mu^{\ a} + \epsilon^a_{\ bc}(e_\mu^{\ b}\omega_\nu^{\ c} + e_\nu^{\ c}\omega_\mu^{\ b} + i\sigma_{\alpha\beta}(\chi_\mu^{\ \alpha}\xi_\nu^{\ \beta} + \xi_\mu^{\ \alpha}\chi_\nu^{\ \beta})] \\
&\quad + J_a[\partial_\mu\omega_\nu^{\ a} - \partial_\nu\omega_\nu^{\ a} + \epsilon^a_{\ bc}\omega_\mu^{\ b}\omega_\nu^{\ c}] \\
&\quad + Q_\alpha[\partial_\mu\chi_\nu^{\ \alpha} - \partial_\nu\chi_\mu^{\ \alpha} - i(\sigma_a)^{\ \alpha}_{\beta}(\omega_\mu^{\ a}\chi_\nu^{\ \beta} - \chi_\mu^{\ \beta}\omega_\nu^{\ a})] \\
&\quad + Q'_\alpha[\partial_\mu\xi_\nu^{\ \alpha} - \partial_\nu\xi_\mu^{\ \alpha} - i(\sigma_a)^{\ \alpha}_{\beta}(\omega_\mu^{\ a}\xi_\nu^{\ \beta} - \xi_\mu^{\ \beta}\omega_\nu^{\ a})]
\end{aligned}
\tag{9}
$$

With these preliminaries,the Chern Simons action for the super Poincaré group can be written as

$$
\begin{aligned}
I_{cs} &= \frac{1}{2}\int_M \{\eta_{bc}[e^b \wedge (2d\omega^c + \epsilon^c_{\ da}\omega^d \wedge \omega^a] \\
&\quad -\epsilon_{\alpha\beta}[\chi^\alpha \wedge (d - i\sigma_a\omega^a)\psi^\beta + \psi^\alpha \wedge (d - i\sigma_a\omega^a)\chi^\beta]\}
\end{aligned}
\tag{10}
$$

This action is invariant under the local gauge transformations

$$\delta A_\mu = \partial_\mu u + i\,[A_\mu, u] \tag{11}$$

where

$$u = \rho^a P_a + \tau^a J_a + \nu^\alpha Q_\alpha + \nu'^\alpha Q'_\alpha \tag{12}$$

To set up a canonical formalism, let the manifold $M$ have the topology $R \times \Sigma$, where $R$ is the real line representing $x^0$ and $\Sigma$ is a two dimensional manifold parametrized by $\{x^i\}$ ,$i = 1, 2$. Then,up to total derivatives,the Chern Simons action (10) reduces to [9]

$$
\begin{aligned}
I_{cs} &= \int dx^0 \int_\Sigma \{-\epsilon^{ij} e_i^a \partial_{x^0}\omega_{aj} + \epsilon^{ij}\epsilon_{\alpha\beta}\chi_i^\alpha \partial_{x^0}\xi_j^\beta \\
&\quad -\eta_{ab}(e_0^a F^b[\omega] + \omega_0^a F^b[e]) + \epsilon_{\alpha\beta}(\chi_0^\alpha F^\beta[\xi] + \xi_0^\alpha F^\beta[\chi])\}
\end{aligned}
\tag{13}
$$

where

$$F^B[A] = \epsilon^{ij} F^B_{ij}[A] \ \ ; \ \ B = (a, \alpha) \tag{14}$$

Analogously to the Chern Simons action for the Poincaré group, the structure of the action (13) indicates that the constraints of the theory are

$$F^a[\omega] = F^a[e] = F^\alpha[\xi] = F^\alpha[\chi] = 0 \tag{15}$$

Thus, the theory is locally trivial, as expected from any pure Chern Simons theory.

To couple (super)sources to this Chern Simons theory, we proceed in a manner similar to the way sources were coupled to the Poincaré Chern Simons theory [5,9,1,2]. It will be recalled that in the latter case a source(particle) was taken to be an irreducible representation of the Poincaré group in 2+1 dimensions[1,2]. Its mass and spin were identified as eigenvalues of the Casimir invariants of the corresponding

Poincaré state. In generalizing this to the supersymmetric case, we take a superparticle(supersource) to be an irreducible representation of the super Poincaré group. From this point of view, a superparticle is an irreducible supermultiplet consisting of several Poincaré states related to each other by the action of the supersymmetric generators. For N=2 super Poincaré group, the lowest dimensional supermultiplet consists of four Poincaré states. In this respect the supermultiplets in $2+1$ and $3+1$ dimensions are similar. We note, however, that in contrast to the $3+1$ dimensional states, the spin of a Poincaré state in 2+1 dimensions is not limited to integer and half integer values. As a result, the spins of states within a supermultiplet are not necessarily integer and half integers. Of course, their spins must still differ from one another by multiples of one-half unit. Here for definiteness we may take the supermultiplet to be the N=2 vector supermultiplet consisting of a spin zero, two spin one-half, and one spin one states. But the action that we write down below for the coupling of an irreducible N=2 supermultiplet to super Chern Simon thoery will be invariant under super Poincaré gauge transformations regardless of the spin content of the supermultiplet.

To couple a supersymmetric source to the N=2 Chern Simons theory, we recall [1,2] that a Poincaré state is characterized by its monentum $p^a = (p^0, \vec{p})$ and its total(Lorentz) angular momentum

$$j^a = \epsilon^a_{\ bc} q^b p^c + s^a \tag{16}$$

where $q^a$ are phase space coordinates canonically conjugate to $p^a$, and $s^a$ determine the intrinsic spin. From these we can construct the Casimir invariants $p^2 = m^2$ and $W^2$ of the Poincaré group, where

$$W = p \cdot j = p \cdot s \ \ ; \ \ W^2 = m^2 s^2 \tag{17}$$

To realize the super Poincaré algebra, we extend the phase space variables $p^a$ and $q^a$ to their supersymmetric form:

$$p^a \longrightarrow (p^a, p^\alpha) \ \ ; \ \ q^a \longrightarrow (q^a, q^\alpha) \tag{18}$$

In terms of these variables, the generators of the super Poincaré algebra take the form

$$\begin{aligned} P_a &= i\partial_a \ \ ; \ \ Q_\alpha = -\partial_\alpha \ \ ; \ \ Q'_\alpha = i(\sigma^a)_{\alpha\beta} q^\beta \partial_a \\ J_a &= \epsilon_{abc} q^b p^c + (\sigma_a)_\alpha^{\ \beta} q^\alpha \partial_\beta + s^a \end{aligned} \tag{19}$$

Then the source action can be written as

$$\begin{aligned} I_s = \int_C d\tau \{ & p_a \partial_\tau q^a - \epsilon_{\alpha\beta} p^\alpha \partial_\tau q^\beta - e^a p_a - \omega^a j_a + i\epsilon_{\alpha\beta} \chi^\alpha p^\beta \\ & -(\sigma \cdot p)_{\alpha\beta} \xi^\alpha q^\beta + \lambda_1 (p^2 - m^2) + \lambda_2 (C_- - ms) \} \end{aligned} \tag{20}$$

where $\tau$ is an invariant parameter along the trajectory $C$. This action is similar to the source action given in reference [9], but it differs from the latter in the definition

of $j_a$ and the choice of the constraint multiplying $\lambda_2$. These features turn out to be crucial in relating the second Casimir invariant, $C_-$, of the superalgebra to the spin content of a supermutiplet [10]. For more than one source, one can add an action of this type for each source.

Under the gauge transformation (12), various quantities in the action transform as follow :

$$\begin{aligned}
\delta e^a_\mu &= \partial_\mu \rho^a + \epsilon^a_{\ bc}(e^b_\mu \tau^c + \omega^b_\mu \rho^c) + i(\sigma^a)_{\alpha\beta}(\chi^\alpha_\mu \nu'^\beta + \xi^\alpha_\mu \nu^\beta) \\
\delta \omega^a_\mu &= \partial_\mu \tau^a + \epsilon^a_{\ bc}\omega^b_\mu \tau^c \\
\delta \chi^\alpha_\mu &= \partial_\mu \nu^\alpha + i(\chi^\beta_\mu \tau^a - \omega^a_\mu \nu^\beta)(\sigma_a)^{\ \alpha}_\beta \\
\delta \xi^\alpha_\mu &= \partial_\mu \nu'^\alpha + i(\xi^\beta_\mu \tau - \omega^a_\mu \nu'_\beta)(\sigma_a)^{\ \alpha}_\beta \\
\delta q^a &= -\rho^a - \epsilon^a_{\ bc}\tau^c q^b - (\sigma^a)_{\alpha\beta}\nu'^\alpha q^\beta \\
\delta q^\alpha &= -i\nu^\alpha + i(\sigma \cdot \tau)^{\ \alpha}_\beta q^\alpha \\
\delta p^a &= -\epsilon^a_{\ bc}\tau^c p^b \\
\delta p^\alpha &= -(\sigma \cdot p)^{\ \alpha}_\beta \nu'^\beta + i(\sigma \cdot \tau)^{\ \alpha}_\beta p^\beta
\end{aligned} \tag{21}$$

It is easy to verify that the combined action $I = I_{cs} + I_s$ is invariant under these infinitesimal gauge transformations.

Next, we turn to the reduction of the two body problem. We use this terminology in the same sense that, e.g. , the familiar two body central force problem can be reduced to an equivalent one body problem. The main difference is that to maintain gauge invariance, in the present case the reduction would have to be carried out in terms of Wilson loops. The Wilson loop for the connection given by Eq. (7) in the representation $R$ of the algebra is given by

$$W_R(C) = Str_R P \exp[i \oint_C A] \tag{22}$$

where *Str* stands for supertrace and P for path ordering. We have identified our superparticles with irreducible representations of the super Poincaré group, indicating that the Casimir invariants of the super Poincaré state are the observables of the superparticle. Since all the gauge invariant observables of a Chern Simons theory are Wilson loops,these Casimir invariants are expressible in terms of Wilson loops. This is in particular true for the equivalent one body state which is also a super Poincaré state. Just as the one body Poincaré state was endowed with momenta $\Pi^a = (\Pi^0, \vec{\Pi})$ and (Lorentz)angular momenta $\Psi^a = (\Psi^0, \vec{\Psi})$ [1,2], we take the one body super Poincaré state to be endowed with charges

$$\Pi^A = (\Pi^a, \Pi^\alpha) \ \ and \ \ \Psi^A = (\Psi^a, \Psi^\alpha) \tag{23}$$

The corresponding Casimir invariants are

$$C_+ = \Pi^a \Pi_a \equiv H^2 \ \ ; \ \ C_- = \Pi^a \Psi_a + \epsilon^{\alpha\beta}\Pi_\alpha \Psi_\beta \tag{24}$$

where $H$ is the mass(Hamiltonian) of the supermultiplet. Being gauge invariant observables of the unbroken supersymmetric gauge theory, they can be expressed in terms of the Wilson loops of the two body system.

In evaluating the invariants $C_+$ and $C_-$ of the super-Poincaré state representing the equivalent one body system in terms of a Wilson loop of actual two body system, it turns out to be technically more convenient to evaluate the corresponding Wilson loop, $\hat{W}_R(C)$, for the super anti-de Sitter algebra $OSp(1|2;R) \times OSp(1|2;R)$ and then obtain the super Poincaré limit by a group contraction. Let $X^A = (X^a, S^X_\alpha)$ and $Y^A = (Y^a, S^Y_\alpha)$ be the generators of the two commuting $OSp(1|2;R)$ subalgebras of the super anti-de Sitter algebra with following non-zero (anti)commutators:

$$\begin{array}{lcl} [X^a, X^b] = -i\epsilon^{abc} X_c & ; & [Y^a, Y^b] = -i\epsilon^{abc} Y_c \\ [X^a, S^X_\alpha] = -(\sigma^a)_\alpha^{\ \beta} S^X_\beta & ; & [Y^a, S^Y_\alpha] = -(\sigma^a)_\alpha^{\ \beta} S^Y_\beta \\ \{S^X_\alpha, S^X_\beta\} = -(\sigma \cdot X)_{\alpha\beta} & ; & \{S^Y_\alpha, S^Y_\beta\} = -(\sigma \cdot Y)_{\alpha\beta} \end{array} \tag{25}$$

Then,let

$$\begin{array}{lcl} \hat{J}_a = X_a + Y_a & ; & \hat{P}_a = \lambda(X_a - Y_a) \\ \hat{Q}_\alpha = \sqrt{\lambda}(S^X_\alpha + S^Y_\alpha) & ; & \hat{Q}'_\alpha = \sqrt{\lambda}(S^X_\alpha - S^Y_\alpha) \end{array} \tag{26}$$

It is easy to verify that these operators satisfy the super anti-de Sitter algebra. The caret on top of these operators as well as the source charges,etc. ,given below indicate that they correspond to the super anti-de Sitter algebra. In the limit $\lambda \longrightarrow 0$, the super Poincaré algebra is recovered. In particular, the Casimir operators $C_\pm$ of the super Poincaré algebra can be recovered in this limit :

$$\hat{C}_\pm = \hat{C}^X \pm \hat{C}^Y \longrightarrow C_\pm \tag{27}$$

where

$$C_+ = P \cdot P \tag{28}$$

$$C_- = P \cdot J + J \cdot P + \epsilon^{\alpha\beta} Q_\alpha Q'_\beta \tag{29}$$

Next, we turn to the computation of $\hat{W}_R(C)$. The source which represents the equivalent one body problem is endowed with super anti-de Sitter charges

$$\hat{\Pi}^A = (\hat{\Pi}^a, \hat{\Pi}^\alpha) \quad ; \quad \hat{\Psi}^A = (\hat{\Psi}^a, \hat{\Psi}^\alpha) \tag{30}$$

These charges are subject to the requirement that after group contraction, $\hat{\Pi}^A \longrightarrow \Pi^A, \hat{\Psi}^A \longrightarrow \Psi^A$ such that

$$C_+ = \eta^{ab} \Pi_a \Pi_b \tag{31}$$

$$C_- = \eta^{ab} \Pi_a \Psi_b + \epsilon^{\alpha\beta} \Pi_\alpha \Psi_\beta \tag{32}$$

Then, the Wilson loop around the source with charges given by (30) will take the form

$$\begin{aligned}
\hat{W}_R(C) &= Str_R P \exp[i(\hat{\Pi}^a \hat{J}_a + \hat{\Psi}^a \hat{P}_a + \hat{\Pi}^\alpha \hat{Q}_\alpha + \hat{\Psi}^\alpha \hat{Q'}_\alpha] \\
&= Str_R P \exp[i(\frac{\hat{Z}_+^A}{2} X_A + \frac{\hat{Z}_-^A}{2} Y_A)]
\end{aligned} \tag{33}$$

where

$$\hat{Z}_\pm^A = (\hat{Z}_\pm^a, \hat{Z}_\pm^\alpha) = (\hat{\Pi}^a \pm \hat{\Psi}^a, \hat{\Pi}^\alpha \pm \hat{\Psi}^\alpha) \tag{34}$$

Just as in the trace computation of Poincaré Chern Simons theory [1,2], the supertrace in (32) can be computed by considering a matrix representation of the superalgebra. The result is

$$W_R(C_0) = (2\cos\frac{|Z_+|}{2} - 1)(2\cos\frac{|Z_-|}{2} - 1) \tag{35}$$

where, after group contraction,

$$|Z_\pm| = [\Pi^a \Pi_a \pm 2(\Pi^a \Psi_a + \epsilon^{\alpha\beta} \Pi_\alpha \Psi_\beta)]^{\frac{1}{2}} \tag{36}$$

To express $|Z_\pm|$ in terms of the properties of the two actual sources, we must set $W_R(C_0)$ equal to a Wilson loop,$W_R(C_{12})$, enclosing these sources. The path $C_{12}$ can be chosen uniquely by the requirement that the equivalent one body problem correctly give the asymptotic observables of the emerging space-time theory, as in the case of the Poincaré theory [1,2]. It turns out that $C_{12}$ is the simple loop enclosing the two sources. Let the two sources have charges $(p_1^A, j_1^A)$ and $(p_2^A, j_2^A)$,respectively. Again we begin with the super anti-de Sitter Wilson loop, $\hat{W}_R(C_{12})$, and obtain $W_R(C_{12})$ by group contraction. We have

$$\hat{W}_R(C_{12}) = Str P \prod_{k=1}^{2} \exp\left[i(\hat{p}_k \cdot \hat{J} + \hat{j}_k \cdot \hat{P} + \hat{p}_k^\alpha \hat{Q}_\alpha + \hat{j}_k^\alpha \hat{Q'}_\alpha)\right] \tag{37}$$

Evaluating this expression and performing the subsequent group contraction, we obtain

$$W_R(C_{12}) = W_x W_y \tag{38}$$

where

$$\begin{aligned}
W_x = \{&1 + 2(\cos\frac{|x_1|}{2} - 1) + 2(\cos\frac{|x_2|}{2} - 1) \\
&-2(|x_1||x_2|)^{-1}(x_1^a \cdot x_2^a - \epsilon_{\alpha\beta} q_1^\alpha q_2^\beta) \sin\frac{|x_1|}{2} \sin\frac{|x_2|}{2} \\
&+(|x_1||x_2|)^{-2}(\cos\frac{|x_1|}{2} - 1)(\cos\frac{|x_2|}{2} - 1) \\
&\times[\, |x_1|^2|x_2|^2 + 4i\epsilon_{abc} x_1^a x_2^b (\sigma)_{\alpha\beta} q_1^\alpha q_2^\beta + (x_1^a)^2 (x_2^a)^2 \\
&\quad -2x_1^a x_2^a \epsilon_{\alpha\beta} q_1^\alpha q_2^\beta + \epsilon_{\alpha\beta}\epsilon_{\gamma\delta} q_1^\alpha q_2^\gamma q_1^\beta q_2^\delta \,]\}
\end{aligned} \tag{39}$$

In this expression,with $k = 1, 2$, and $x^A = (x^a, q^\alpha)$, we have

$$x_k^a = p_k^a + j_k^a \quad ; \quad q_k^\alpha = p_k^\alpha + j_k^\alpha \tag{40}$$

The factor $W_y$ has exactly the same structure as $W_x$, i.e., $x^A \longrightarrow y^A = (y^a, q'^\alpha)$,where

$$y_k^a = p_k^a - j_k^a \quad ; \quad q_k'^\alpha = p_k^\alpha - j_k^\alpha \tag{41}$$

Having evaluated the Wilson loops, we can now set

$$W_R(C_0) = W_R(C_{12}) \tag{42}$$

to obtain the Casimir invariants $C_\pm$ of the supermultiplet representing the equivalent one body formalism. Since $C_+$ and $C_-$ are independent, consider first $C_+$ given by (24). To evaluate it, we set the charges $j^A$ and $p^\alpha$, which contribute to $C_-$ but not to $C_+$, equal to zero. Then, from (35), (39), and (41), we obtain

$$\cos\frac{H}{2} = \cos\frac{m_1}{2}\cos\frac{m_2}{2} - \frac{p_1 \cdot p_2}{m_1 m_2}\sin\frac{m_1}{2}\sin\frac{m_2}{2} \tag{43}$$

Not surprisingly, this is exactly the same expression that was obtained for the mass of the equivalent one body state in the Poincaré Chern Simons theory[1,2]. This is as it should be because the quantity $H$ is also the common mass of the Poincaré states within our supermultiplet and could be evaluated from the two body system independently of the supersymmetry. Making use of this *a posteriori* knowledge, the extraction of equation (43) from (35), (39), and (41) may viewed as a test of the correctness of our reduction formalism.

Having evaluated $H$ ,i.e., $C_+$, from equation (43), we can substitute it into (35) and use (39) and (49) to obtain an exact, albeit implicit expression for $C_-$. A more detailed discussion of the properties of the supersymmetric two body system and its applications will be given elsewhere[10].

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